Int. J. Solids Structures, Vol. 17, No. 12, pp. 1159–1178, 1981
Printed in Great Britain.

0020-7683/81/121159-20$02.00/0
Pergamon Press Ltd.

A TWO-DIMENSIONAL THEORY FOR PIEZOELECTRIC LAYERS USED IN ELECTRO-MECHANICAL TRANSDUCERS—I
DERIVATION

N. BUGDAYCI and D. B. BOGY
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, U.S.A.

(*Received 5 December* 1980)

Abstract—An Nth order two-dimensional theory is derived for strongly coupled piezoelectric layers that is suitable for use in analyzing electro-mechanical transducer response in circumstances where the field variations over the transducer face are important. Specific formulae are given for the widely used ceramic PZT-5.

The first few branches of the dispersion curves for PZT-5 are obtained numerically for both the three-dimensional theory and the approximate two-dimensional theory. The correspondence of these curves at long wavelengths is used to determine a single correction factor. The face boundary conditions accommodate unknown traction and voltage so that the theory can be used in transducer applications.

1. INTRODUCTION

This series of two papers aims to develop the theory necessary for analyzing electro-mechanical transducers in cases when the mechanical loading over the face of the active element is not spatially uniform. For our purposes the transducer is considered to be a piezoelectric plate element made of polarized ferroelectric ceramic. This plate is an element in an electrical circuit with known input impedance and it has its two faces coated with thin electrodes. The faces are thus constrained to be electrical equi-potential surfaces. The mechanical effects of the electrode coating are neglected, and we will not be considering here specific transducer housing designs and mounting techniques. Part I is concerned with the general theoretical development of the approximate 2-D theory and Part II contains some specific solutions relevant to transducer analysis.

First, we summarize the 3-D field equations of linear piezoelectricity theory. Resulting from this is a set of differential equations that determine the motion in the layer. It is found that significant information about the motion can be extracted from the structure of these equations without explicitly solving them. Explicit solutions have been previously obtained, however, for the special case of an infinite layer with vanishing tractions and vanishing electric potential at the two faces[1]. These special case solutions prove useful in the evaluation of approximate 2-D theories that are developed to deal with the general case of a bounded plate with non-trivial face and edge boundary conditions.

Employing a general procedure introduced by Mindlin[2], but using trigonometric series expansions instead of power series expansions, Lee and his students[3–5] have deduced 2-D equations of motion of successively higher orders of approximation for isotropic elastic, anisotropic elastic and piezoelectric crystal plates. In all these applications, no more than two correction factors were utilized to produce from their approximate theories, the "exact" cut-off frequencies given by the 3-D theory. Hence, they have achieved very close matching of the dispersion curves of the approximate and 3-D theories for long wavelengths.

Among the references cited above, the paper by Syngellakis and Lee[5] comes closest to having direct applicability to the subject of the present investigation. In that paper, 2-D approximate field equations are derived for a piezoelectric plate with m-monoclinic symmetry. However, some of the inhomogeneous terms in these differential equations involve the magnitudes of the normal components of the electrical displacement vector at the two faces of the plate. This theory is directly applicable to problems in which these magnitudes are specified as a function of the in-plane coordinates. This occurs, for example, when the faces of the plate are in contact with non-conducting media in which case the current across the faces and the normal components of the electric displacement vector at the faces vanish. However, the usefulness of this theory is severely diminished in problems where these functions are unknown.

This is the difficulty encountered in the case of a transducer in which the faces of the plate are coated with highly conductive electrodes. In dealing with such problems, the development of a new set of approximate field equations that incorporate the modified face boundary conditions in their derivation seems to be called for.

Such an approximate 2-D theory is developed here in Part I for polarized ferroelectric ceramic plates at the faces of which the electric potential is specified as a function of time only. The development is based on trigonometric expansions in the thickness-coordinate of the displacement components, as found in the references cited above, along with a new trigonometric expansion for the electric scalar potential function that satisfies specified boundary values at the two faces. From these expansions 2-D field equation as well as 2-D kinetic and internal energy densities and electric enthalpy density expressions are deduced and set forth for the truncated system of arbitrarily large order N. For first and second order approximations, the dispersion curves for an infinite plate for real and imaginary wave numbers are examined and compared with corresponding solutions of the exact frequency equations of the 3-D theory. The ranges of frequency and wave number for which these lower order theories are applicable, are thus determined.

In Part II the special cases of the plain strain motion of an infinite layer with rectangular cross section and the axisymmetric motion of a circular layer are studied. For each case the method of obtaining the closed-form analytical solution corresponding to the Nth order theory is presented in detail. Both steady time-harmonic solutions and transient solutions for initial-boundary value problems are given. In the axisymmetric case the output voltage as a function of time resulting from the sudden release of a central point load is computed for a simply supported circular plate.

2. THREE-DIMENSIONAL THEORY

The 3-D linear theory of piezoelectricity is presented in Ref. [6] and is summarized by the following system of 22 field equations in 22 unknown variables that are functions of position x_j and time t.

$$T_{ij,i} = \rho \ddot{u}_j \tag{2.1}$$

$$D_{i,i} = 0 \tag{2.2}$$

$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k \tag{2.3}$$

$$D_i = e_{ikl} S_{kl} + \epsilon_{ik} E_k \tag{2.4}$$

$$S_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{2.5}$$

$$E_k = -\phi_{,k}. \tag{2.6}$$

The lower case Latin indices range over 1, 2, 3; summation of repeated indices is implied; a comma denotes partial differentiation and a superposed dot indicates differentiation with respect to time.

The terms that appear in these equations are Cartesian components of tensors as listed below:

- T_{ij} stress tensor
- D_i electric displacement vector
- S_{ij} strain tensor
- E_i electric field intensity vector
- u_i displacement vector
- ϕ electric scalar potential

ρ mass density
c_{ijkl} elastic material tensor
e_{ikl} piezoelectric material tensor
ϵ_{ij} dielectric material tensor.

We assume that the material is homogeneous in which case ρ, c_{ijkl}, e_{ikl} and ϵ_{ij} are constants.

Equations (2.1) are the local stress equations in the absence of body forces. The condition (2.2) on the electric displacement vector involves the assumption of a quasistatic electric field and is the electrical equivalent of the mechanical equations of motion (2.1). Equations (2.3) and (2.4) are the constitutive equations relating stress and electric displacement to strain and electric field intensity. Through the material constants these equations characterize the behaviour of the piezoelectric material under consideration. Finally, eqns (2.5) and (2.6) give the definitions of mechanical strain and electric field intensity in terms of the displacement components and electric potential, respectively.

We note here the generally valid symmetry restrictions on the material constants that appear in the constitutive equations.

$$\left.\begin{array}{l} c_{ijkl} = c_{jikl} = c_{klij} \\ e_{kij} = e_{kji};\ \epsilon_{ij} = \epsilon_{ji} \end{array}\right\}. \tag{2.7}$$

By successive substitutions, the original system of eqns (2.1)–(2.6) can be reduced to a set of four equations in the four unknown variables u_i and ϕ:

$$c_{ijkl}u_{k,li} + e_{kij}\phi_{,ki} = \rho\ddot{u}_j \tag{2.8}$$

$$e_{kij}u_{i,jk} - \epsilon_{ij}\phi_{,ij} = 0. \tag{2.9}$$

In a linear piezoelectric medium, the internal energy density U, the kinetic energy K and the electric enthalpy density H are given by the following expressions:

$$U = \frac{1}{2}c_{ijkl}S_{ij}S_{kl} + \frac{1}{2}\epsilon_{ij}E_iE_j \tag{2.10}$$

$$K = \frac{1}{2}\rho\dot{u}_j\dot{u}_j \tag{2.11}$$

$$H = U - E_iD_i \tag{2.12}$$

or

$$H = \frac{1}{2}c_{ijkl}S_{ij}S_{kl} - \frac{1}{2}\epsilon_{ij}E_iE_j - e_{ijk}E_iS_{jk} \tag{2.13}$$

so that

$$T_{ij} = \frac{\partial H}{\partial S_{ij}};\ D_i = -\frac{\partial H}{\partial E_i}. \tag{2.14}$$

Also, from (2.10), (2.3) and (2.4), an energy balance equation can be derived:

$$\dot{U} = T_{ij}\dot{S}_{ij} + E_i\dot{D}_i. \tag{2.15}$$

U and K are assumed to be positive–definite quadratic functions of S_{ij}, E_i and $\dot{u}_j$. This is used in deducing a uniqueness theorem for linear piezoelectricity analogous to Neumann's theorem in elasticity. It is thus proved that a unique solution to the system of eqns (2.1)–(2.6) can be insured by specifying at each point of the body at time t_o, the quantities u_j, $\dot{u}_j$ and E_j and by specifying at each point of the surface of the body at all times one member of each of the products,

$$t_1u_1,\ t_2u_2,\ t_3u_3,\ \sigma\phi,$$

where

$$t_i = T_{ij}n_j;\ \sigma = D_in_i. \tag{2.16}$$

In eqns (2.16), t_i and σ are the surface tractions and surface charge, respectively, and n_i are the components of the unit outward normal vector on the surface.

The eqns (2.1) and (2.2) and the surface boundary conditions can be alternatively derived from the following variational principle for a piezoelectric continuum. In a region V bounded by a surface S,

$$\delta\int_{t_o}^{t_1}\mathrm{d}\tau\int_V (K-H)\,\mathrm{d}V + \int_{t_o}^{t_1}\mathrm{d}\tau\int_S (t_j\delta u_j + \sigma\delta\phi)\,\mathrm{d}S = 0 \tag{2.17}$$

where δu_j and $\delta\phi$ are required to vanish at t_0, t_1. The variational principle represented by this equation will be used in the derivation of 2-D approximate theories in the next chapter.

We will apply the piezoelectric theory described above to the special case of a polarized ferroelectric ceramic, which has the symmetry of a hexagonal crystal in class C_{6v} or 6 mm [6, 7]. The restrictions on the elastic, piezoelectric and dielectric constants of a material possessing this symmetry (with the symmetry around the x_2-axis) are expressed by the following arrays:

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}$$

where

$$c_{55} = \frac{1}{2}(c_{11} - c_{13}) \tag{2.18}$$

$$[e_{iJ}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & e_{16} \\ e_{21} & e_{22} & e_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{16} & 0 & 0 \end{bmatrix}$$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{11} \end{bmatrix}$$

In eqns (2.18), the usual compressed subscript notation is used where capital Latin indices replace pairs of lower case Latin indices. Accordingly, these capital indices take values 1–6.

We treat here only ferroelectric ceramic layers that are polarized in the thickness direction. We let the coordinate system to which such a layer is referred be as shown in Fig. 1 with its origin located in the middle plane and with the x_2-axis along the thickness direction. All the points on the faces thus have $x_2 = \pm b$ where $2b$ is the thickness of the layer. The middle plane intersects the right cylindrical edge boundary B of the layer in a curve C that lies on the $x_1 - x_3$ plane. It can be verified that because of the symmetry of the material around the poling direction, the tensor components c_{ijkl}, e_{kij} and ϵ_{ij} remain invariant under coordinate transformations that leave the x_2-direction unchanged.

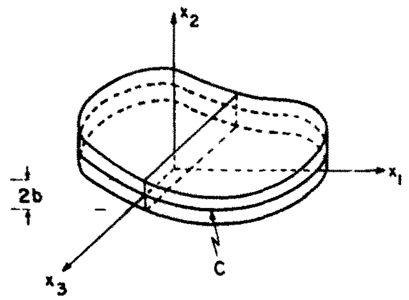

Fig. 1. Piezoelectric layer and coordinate system.

Next we formulate a set of boundary conditions appropriate for the analysis of the transducer problem. First, we specify at the two faces of the layer the tractions and the values of the electric potential:

$$\begin{aligned} T_{2j}(x_1, b, x_3, t) &= T^+_{2j}(x_1, x_3, t) \\ T_{2j}(x_1, -b, x_3, t) &= T^-_{2j}(x_1, x_3, t) \\ \phi(x_1, b, x_3, t) &= \phi^+(t) \\ \phi(x_1, -b, x_3, t) &= \phi^-(t) \end{aligned} \tag{2.19}$$

where T^+_{2j}, T^-_{2j}, ϕ^+ and ϕ^- are prescribed functions. For a layer that is not infinite in all directions certain edge boundary conditions are needed as a requirement of a well-posed problem. Accordingly, we specify the tractions at the right cylindrical edge boundary B and require vanishing current across this boundary:

$$\left.\begin{aligned} T_{j\gamma}n_\gamma &= \bar{t}_j(\underset{\sim}{x}, t) \\ D_\gamma n_\gamma &= 0 \end{aligned}\right\} \text{for } \underset{\sim}{x} \in B. \tag{2.20}$$

As initial conditions we specify u_j, $\dot{u}_j$ and ϕ throughout the body V at time t_o. It can be shown[8] that this initial-boundary value problem completely uncouples into flexural and extensional modes of motion.

3. APPROXIMATE THEORIES

The approximate 2-D theories are derived here by substituting truncated trigonometric series expansions of displacement components and the electric scalar potential into the variational principle stated in eqn (2.17). Hence our first task is to produce for these functions the appropriate infinite trigonometric series expansions on which the truncations are to be made.

For a bounded or an unbounded plate referred to a coordinate system as in Fig. 1, it is possible to make the following Fourier cosine series expansion of the dependence on the thickness coordinate x_2 of the displacement component functions:

$$u_j(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} u_j^{(n)}(x_1, x_3, t) \cos \frac{n\pi}{2}(1-\psi) \tag{3.1}$$

where

$$\psi = \frac{x_2}{b}.$$

For the electric scalar potential, whose values at the faces of the plate are to be prescribed by boundary conditions, a Fourier sine series expansion is made for the difference between the actual electric potential and a linear approximation to it that satisfies the two specified boundary values. Since this "difference function" must vanish at the faces, the suitability of a Fourier sine series expansion is evident. Thus, we have:

$$\phi(x_1, x_2, x_3, t) = A(t) + B(t)\psi + \sum_{n=0}^{\infty} \left\{ \phi^{(n)}(x_1, x_3, t) \sin \frac{n\pi}{2}(1-\psi) \right\} \tag{3.2}$$

and we can take $\phi^{(o)} = 0$ without any loss in generality. The coefficients of the linear approximation, A and B, are determined by the specified boundary values of the electric potential,

$$\phi(x_1, b, x_3, t) = \phi^+(t); \phi(x_1, -b, x_3, t) = \phi^-(t) \tag{3.3}$$

in the following way:

$$A = \frac{1}{2}(\phi^+ + \phi^-), \; B = \frac{1}{2}(\phi^+ - \phi^-). \tag{3.4}$$

The process of choosing the relatively more important terms of the series in eqns (3.1) and (3.2) is the truncation process, and it determines the order of the resulting approximate theory. The lowest order theory, which is labeled here as the zero-order theory, involves a special truncation procedure and a number of additional assumptions that do not have equivalents in the treatment of higher order theories. This theory is developed in [8] and found to be unsuitable for the purposes of transducer analysis under the conditions of interest. For this reason the Nth order theory ($N \geq 1$) is presented here.

Nth order theory (N $\geq$ 1)

In the Nth order theory the truncation procedure consists of setting to zero in (3.1) and (3.2) all terms of order higher than N:

$$\left.\begin{matrix} u_j^{(n)} = 0 \\ \phi^{(n)} = 0 \end{matrix}\right\} \text{ for } n > N. \tag{3.5}$$

From this we get

$$u_j = \sum_{n=0}^{N} u_j^{(n)} \cos \frac{n\pi}{2}(1-\psi), \tag{3.6}$$

and

$$\phi = A + B\psi + \sum_{n=1}^{N} \phi^{(n)} \sin \frac{n\pi}{2}(1-\psi). \tag{3.7}$$

Substituting (3.6) and (3.7) into (2.5) and (2.6), we obtain

$$S_{ij} = \sum_{n=0}^{N} \left\{ S_{ij}^{(n)} \cos \frac{n\pi}{2}(1-\psi) + \bar{S}_{ij}^{(n)} \sin \frac{n\pi}{2}(1-\psi) \right\}$$

$$E_i = \sum_{n=0}^{N} \left\{ E_i^{(n)} \cos \frac{n\pi}{2}(1-\psi) + \bar{E}_i^{(n)} \sin \frac{n\pi}{2}(1-\psi) \right\} \tag{3.8}$$

where $S_{ij}^{(n)}$, $\bar{S}_{ij}^{(n)}$, the nth order components of 2-D strain and $E_i^{(n)}$, $\bar{E}_i^{(n)}$, the nth order components of 2-D electric field intensities are defined as follows:

$$
\begin{aligned}
S_{ij}^{(n)} &= \frac{1}{2}(u_{i,j}^{(n)} + u_{j,i}^{(n)}) \\
\bar{S}_{ij}^{(n)} &= \frac{n\pi}{4b}(\delta_{2i}u_j^{(n)} + \delta_{2j}u_i^{(n)}) \\
E_i^{(n)} &= -\delta_{no}\delta_{2i}B/b + \delta_{2i}\frac{n\pi}{2b}\phi^{(n)} \\
\bar{E}_i^{(n)} &= -\phi_{,i}^{(n)}
\end{aligned}
\tag{3.9}
$$

in which δ_{rs} represents the Kronecker delta:

$$
\delta_{rs} = \begin{cases} 1 \text{ for } r = s \\ 0 \text{ for } r \neq s. \end{cases}
$$

Next the 2-D stress and electric displacement components are defined by

$$
\begin{aligned}
T_{ij}^{(n)} &= \int_{-1}^{1} T_{ij}\cos\frac{n\pi}{2}(1-\psi)\,d\psi \\
\bar{T}_{ij}^{(n)} &= \int_{-1}^{1} T_{ij}\sin\frac{n\pi}{2}(1-\psi)\,d\psi \\
D_i^{(n)} &= \int_{-1}^{1} D_i\cos\frac{n\pi}{2}(1-\psi)\,d\psi \\
\bar{D}_i^{(n)} &= \int_{-1}^{1} D_i\sin\frac{n\pi}{2}(1-\psi)\,d\psi.
\end{aligned}
\tag{3.10}
$$

Substitution of (3.8) into (2.3) and (2.4), then, in turn, into (3.10) yields the 2-D nth order constitutive relations:

$$
\begin{aligned}
T_{ij}^{(n)} &= (1+\delta_{no})(c_{ijkl}S_{kl}^{(n)} - e_{kij}E_k^{(n)}) + \sum_{m=1}^{N}\{A_{mn}(c_{ijkl}\bar{S}_{kl}^{(m)} - e_{kij}\bar{E}_k^{(m)})\} \\
D_i^{(n)} &= (1+\delta_{no})(e_{ijk}S_{jk}^{(n)} + \epsilon_{ij}E_j^{(n)}) + \sum_{m=1}^{N}\{A_{mn}(e_{ijk}\bar{S}_{jk}^{(m)} + \epsilon_{ij}\bar{E}_j^{(m)})\} \\
\bar{T}_{ij}^{(n)} &= \sum_{m=0}^{N}\{A_{nm}(c_{ijkl}S_{kl}^{(m)} - e_{kij}E_k^{(m)})\} + c_{ijkl}\bar{S}_{kl}^{(n)} - e_{kij}\bar{E}_k^{(n)} \\
\bar{D}_i^{(n)} &= \sum_{m=0}^{N}\{A_{nm}(e_{ijk}S_{jk}^{(m)} + \epsilon_{ij}E_j^{(m)}) + e_{ijk}\bar{S}_{jk}^{(n)} + \epsilon_{ij}\bar{E}_j^{(n)}
\end{aligned}
\tag{3.11}
$$

in which

$$
A_{mn} = \int_{-1}^{1}\sin\frac{m\pi}{2}(1-\psi)\cos\frac{n\pi}{2}(1-\psi)\,d\psi
$$

so that

$$
A_{mn} = \begin{cases} 0 & \text{for } m+n \text{ even} \\ \dfrac{4m}{(m^2-n^2)\pi} & \text{for } m+n \text{ odd.} \end{cases}
\tag{3.12}
$$

In deriving (3.11) we have also made use of the following identities:

$$
\begin{aligned}
\int_{-1}^{1}\sin\frac{n\pi}{2}(1-\psi)\sin\frac{m\pi}{2}(1-\psi)\,d\psi &= \delta_{nm} - \delta_{no}\delta_{mo} \\
\int_{-1}^{1}\cos\frac{n\pi}{2}(1-\psi)\cos\frac{m\pi}{2}(1-\psi)\,d\psi &= \delta_{nm} + \delta_{no}\delta_{mo}
\end{aligned}
\tag{3.13}
$$

We proceed next to the consideration of energy and energy-related quantities. In this regard, we define 2-D kinetic energy, electric enthalphy and internal energy densities as follows:

$$\bar{K}=\int_{-1}^{1} K\,d\psi;\ \bar{H}=\int_{-1}^{1} H\,d\psi;\ \bar{U}=\int_{-1}^{1} U\,d\psi. \tag{3.14}$$

Use of (2.11) in the first of (3.14) with (3.6) and (3.13) yields

$$\bar{K}=\frac{1}{2}\rho\sum_{n=0}^{N}(1+\delta_{no})\dot{u}_j^{(n)}\dot{u}_j^{(n)}. \tag{3.15}$$

For the internal energy and electric enthalphy densities, we first use the constitutive equations (2.3) and (2.4) in (2.10) and (2.12) to get

$$U=\frac{1}{2}(T_{ij}S_{ij}+E_iD_i) \tag{3.16}$$

$$H=\frac{1}{2}(T_{ij}S_{ij}-E_iD_i). \tag{3.17}$$

Substituting next (3.8) into (3.16) and (3.17), then, in turn, into the second and third of (3.14) and using also (3.10), we arrive at the results:

$$\bar{U}=\frac{1}{2}\sum_{n=0}^{N}\{T_{ij}^{(n)}S_{ij}^{(n)}+\bar{T}_{ij}^{(n)}\bar{S}_{ij}^{(n)}+E_i^{(n)}D_i^{(n)}+\bar{E}_i^{(n)}\bar{D}_i^{(n)}\} \tag{3.18}$$

$$\bar{H}=\frac{1}{2}\sum_{n=0}^{N}\{T_{ij}^{(n)}S_{ij}^{(n)}+\bar{T}_{ij}^{(n)}\bar{S}_{ij}^{(n)}-E_i^{(n)}D_i^{(n)}-\bar{E}_i^{(n)}\bar{D}_i^{(n)}\}. \tag{3.19}$$

These expressions for $\bar{U}$ and $\bar{H}$ can be expanded by the use of (3.11) so as to fully show their dependence on the 2-D strain and electric field intensity components. For $\bar{H}$, in particular, the amplified expression takes the form

$$\begin{aligned}2\bar{H}=c_{ijkl}&\left\{\sum_{n=0}^{N}[(1+\delta_{no})S_{ij}^{(n)}S_{kl}^{(n)}+\bar{S}_{ij}^{(n)}\bar{S}_{kl}^{(n)}]+2\sum_{m=0}^{N}\sum_{n=0}^{N}[B_{nm}S_{ij}^{(m)}\bar{S}_{kl}^{(n)}]\right\}\\ -2e_{kij}&\left\{\sum_{n=0}^{N}[(1+\delta_{no})S_{ij}^{(n)}E_k^{(n)}+\bar{S}_{ij}^{(n)}\bar{E}_k^{(n)}]+\sum_{m=0}^{N}\sum_{n=0}^{N}[B_{nm}(S_{ij}^{(m)}\bar{E}_k^{(n)}+E_k^{(m)}\bar{S}_{ij}^{(n)})]\right\}\\ -\epsilon_{kl}&\left\{\sum_{n=0}^{N}[(1+\delta_{n0})E_k^{(n)}E_l^{(n)}+\bar{E}_k^{(n)}\bar{E}_l^{(n)}]+2\sum_{m=0}^{N}\sum_{n=0}^{N}[B_{nm}E_k^{(m)}\bar{E}_l^{(n)}]\right\}.\end{aligned} \tag{3.20}$$

We note that $\bar{H}$ is a quadratic, homogeneous function of the 2-D strain and electric field components, $S_{ij}^{(n)}$, $\bar{S}_{ij}^{(n)}$, $E_i^{(n)}$, $\bar{E}_i^{(n)}$.

A correction factor α has been introduced into (3.20) through the following definition of B_{mn}:

$$B_{10}=\alpha A_{10} \qquad B_{mn}=A_{mn},\ (m,n)\neq(1,0). \tag{3.21}$$

The value of α is to be determined later by comparison of certain dispersion curve branches calculated from the 2-D and "exact" 3-D theories.

Except for the consideration of the correction factor, α, it can be directly verified from (3.21) that the following relations are satisfied:

$$T_{ij}^{(n)}=\frac{\partial\bar{H}}{\partial S_{ij}^{(n)}};\quad \bar{T}_{ij}^{(n)}=\frac{\partial\bar{H}}{\partial\bar{S}_{ij}^{(n)}}$$

$$D_i^{(n)}=-\frac{\partial\bar{H}}{\partial E_i^{(n)}};\qquad \bar{D}_i^{(n)}=-\frac{\partial\bar{H}}{\partial\bar{E}_i^{(n)}}. \tag{3.22}$$

Two dimensional constitutive relations derived from (3.22) and (3.20) will be used in the remainder of this paper. Replacement of A_{mn} by B_{mn} in (3.11) produces such modified constitutive relations.

The development up to this point of the approximate Nth order theory lays the groundwork for the application of the variational principle of (2.17) for the purpose of deriving the stress equations of motion and charge equations of the 2-D theory.

Using (3.14) and taking the variation inside the integral signs, we rewrite (2.17) as:

$$b\int_{t_o}^{t}\mathrm{d}\tau\int_A \delta(\bar{K}-\bar{H})\,\mathrm{d}A+\int_{t_o}^{t}\mathrm{d}\tau\int_S (t_j\delta u_j+\sigma\delta\phi)\,\mathrm{d}S=0, \tag{3.23}$$

in which A represents the area of the middle plane, and S represents the total boundary of the layer as a 3-D body, that is, the union of upper and lower face boundaries S^+ and S^- and the edge boundary B.

We now consider (3.23) term by term and after some lengthy manipulations obtain

$$b\int_{t_o}^{t}\mathrm{d}\tau\int_A \delta\bar{K}\,\mathrm{d}A=-b\int_{t_o}^{t}\mathrm{d}\tau\int_A\left\{\rho\sum_{n=0}^{N}(1+\delta_{no})\ddot{u}_j^{(n)}\delta u_j^{(n)}\right\}\mathrm{d}A \tag{3.24}$$

in which we have used the fact that $\delta u_j^{(n)}$ vanishes at t_0 and t.

$$\begin{aligned}-b\int_{t_o}^{t}\mathrm{d}\tau\int_A \delta\bar{H}\,\mathrm{d}A=&-b\int_{t_o}^{t}\mathrm{d}\tau\int_C\sum_{n=0}^{N}\{T_{i\gamma}^{(n)}n_\gamma\delta u_i^{(n)}+\bar{D}_\gamma^{(n)}n_\gamma\delta\phi^{(n)}\}\,\mathrm{d}C\\&+b\int_{t_o}^{t}\mathrm{d}\tau\int_A\sum_{n=0}^{N}\left\{\left(T_{ij,j}^{(n)}-\frac{n\pi}{2b}\bar{T}_{i2}^{(n)}\right)\delta u_i^{(n)}+\left(\bar{D}_{i,i}^{(n)}+\frac{n\pi}{2b}D_2^{(n)}\right)\delta\phi^{(n)}\right\}\mathrm{d}A,\end{aligned} \tag{3.25}$$

in the derivation of which we have used (3.22) and (3.9), the symmetry of $T_{ij}^{(n)}$ and $\bar{T}_{ij}^{(n)}$, and the divergence theorem.

$$\begin{aligned}\int_{t_o}^{t}\mathrm{d}\tau\int_S(t_j\delta u_j+\sigma\delta\phi)\,\mathrm{d}S=&\int_{t_o}^{t}\mathrm{d}\tau\int_A\sum_{n=0}^{N}F_i^{(n)}\delta u_i^{(n)}\,\mathrm{d}A\\&+b\int_{t_o}^{t}\mathrm{d}\tau\int_C\sum_{n=0}^{N}(t_j^{(n)}\delta u_j^{(n)}+\bar{\sigma}^{(n)}\delta\phi^{(n)})\,\mathrm{d}C\end{aligned} \tag{3.26}$$

in the derivation of which we have used $S=S^+US^-UB$ as well as (3.6), (3.7) and the fact that $\delta\phi=0$ on S^+ and S^-. Also, the quantities $F_i^{(n)}$, $t_j^{(n)}$, and $\bar{\sigma}^{(n)}$ are defined by

$$\begin{aligned}F_i^{(n)}(x_1,x_3,t)&=T_{i2}|_{x_2=b}-(-1)^nT_{i2}|_{x_2=-b}\\t_j^{(n)}&=\int_{-1}^{1}t_j\cos\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi\\\bar{\sigma}^{(n)}&=\int_{-1}^{1}\sigma\sin\frac{n\pi}{2}(1-\psi)\,\mathrm{d}\psi.\end{aligned} \tag{3.27}$$

The definitions of $t_j^{(n)}$ and $\bar{\sigma}^{(n)}$ hold inside A as well as on the boundary of A.

Substituting (3.24)–(3.27) back into (3.23) we obtain a variational principle in terms of the variations of the 2-D components, $\delta u_i^{(n)}$ and $\delta\phi^{(n)}$:

$$\begin{aligned}&b\int_{t_o}^{t}\mathrm{d}\tau\int_A\sum_{n=0}^{N}\left\{\left[T_{ij,j}^{(n)}-\frac{n\pi}{2b}\bar{T}_{i2}^{(n)}+\frac{1}{b}F_i^{(n)}\right.\right.\\&\left.\left.-\rho(1+\delta_{no})\ddot{u}_i^{(n)}\right]\delta u_i^{(n)}+\left(\bar{D}_{i,i}^{(n)}+\frac{n\pi}{2b}D_2^{(n)}\right)\delta\phi^{(n)}\right\}\mathrm{d}A\\&+b\int_{t_o}^{t}\mathrm{d}\tau\int_C\sum_{n=0}^{N}\{(t_i^{(n)}-T_{i\gamma}^{(n)}n_\gamma)\delta u_i^{(n)}\\&+(\bar{\sigma}^{(n)}-\bar{D}_\gamma^{(n)}n_\gamma)\delta\phi^{(n)}\}\,\mathrm{d}C=0.\end{aligned} \tag{3.28}$$

Since the variations $\delta u_i^{(n)}$ and $\delta\phi^{(n)}$ are arbitrary inside A, we obtain from this the 2-D stress equations of motion:

$$T_{ij,i}^{(n)} - \frac{n\pi}{2b}\bar{T}_{i2}^{(n)} + \frac{1}{b}F_i^{(n)} = \rho(1+\delta_{no})\ddot{u}_i^{(n)} \qquad n = 0, 1, \ldots, N \tag{3.29}$$

and the 2-D charge equations:

$$\bar{D}_{i,i}^{(n)} + \frac{n\pi}{2b}D_2^{(n)} = 0, \qquad n = 1, 2, \ldots, N. \tag{3.30}$$

Also, on the boundary C of area A we conclude:

(a) Either $\delta u_i^{(n)}$ is arbitrary and $t_i^{(n)} = T_{i\gamma}^{(n)}n_\gamma$ or $u_i^{(n)}$ is prescribed and $\delta u_i^{(n)}$ is zero for $n = 0, 1, \ldots, N$.

(b) Either $\delta\phi^{(n)}$ is arbitrary and $\bar{\sigma}^{(n)} = \bar{D}_\gamma^{(n)}n_\gamma$, or $\phi^{(n)}$ is prescribed and $\delta\phi^{(n)}$ is zero, for $n = 1, 2, \ldots, N$.

We see that in addition to the stress equations of motion and charge equations, the necessary edge boundary conditions have resulted from the variational principle. Thus, the formulation of the 2-D field equations for the approximate Nth order theory is complete. Equations (3.29), (3.30) and (3.11) with A_{mn} replaced by B_{mn} and (3.9) form a system of field equations analogous to the set of eqns (2.1)–(2.6) of the 3-D theory. The variables involved in the 2-D system are listed in Table 1.

The total number of variable components listed in Table 1 adds up to $35N + 18$ for Nth order theory. The total number of field equations used in the Nth order system can also be counted to be $35N + 18$.

As in the 3-D theory this large system can be reduced by successive substitutions to a system of $4N + 3$ field equations in $4N + 3$ electrical potential and displacement components, $\phi^{(n)}$, $u_i^{(n)}$:

$$\begin{aligned}(1+\delta_{no})c_{ijkl}u_{k,li}^{(n)} &+ \sum_{m=1}^{N}\{B_{mn}e_{kij}\phi_{,ki}^{(m)}\} + \frac{\pi}{2b}\sum_{m=0}^{N}\{mB_{mn}c_{ij2l}u_{l,i}^{(m)} - nB_{nm}c_{2jkl}u_{k,l}^{(m)}\}\\ &- \frac{n\pi}{2b}(e_{2ij}+e_{i2j})\phi_{,i}^{(n)} - \left(\frac{n\pi}{2b}\right)^2 c_{2j2l}u_l^{(n)} + \left(\frac{\pi}{2b}\right)^2\sum_{m=1}^{N}\{mnB_{nm}e_{22j}\phi^{(m)}\} - (1+\delta_{no})\rho\ddot{u}_j^{(n)}\\ &= \frac{-1}{b}F_j^{(n)} + \frac{n\pi}{2b}e_{k2j}B_{no}\delta_{2k}B/b.\end{aligned} \tag{3.31}$$

Table 1

Variable	Range of Order, n	Total Number of Components
$T_{ij}^{(n)}$	n=0,1,...,N	6(N+1)
$\bar{T}_{i2}^{(n)}$	n=1,...,N	3N
$D_2^{(n)}$	n=1,...,N	N
$\bar{D}_i^{(n)}$	n=1,...,N	3N
$S_{ij}^{(n)}$	n=0,1,...,N	6(N+1)
$\bar{S}_{ij}^{(n)}$	n=1,...,N	6N
$E_i^{(n)}$	n=0,1,...,N	3(N+1)
$\bar{E}_i^{(n)}$	n=1,...,N	3N
$u_i^{(n)}$	n=0,1,...,N	3(N+1)
$\phi^{(n)}$	n=1,...,N	N

for $n = 0, 1, \ldots, N$: $j = 1, 2, 3$

$$\sum_{m=0}^{N} \{B_{nm} e_{ijk} u_{j,ki}^{(m)}\} - \epsilon_{ij} \phi_{,ij}^{(n)} + \frac{n\pi}{2b} (e_{i2k} + e_{2ki}) u_{k,i}^{(n)}$$
$$+ \frac{\pi}{2b} \sum_{m=1}^{N} \{\epsilon_{i2}(mB_{nm} - nB_{mn}) \phi_{,i}^{(m)}\} + \left(\frac{\pi}{2b}\right)^2 \sum_{m=0}^{N} \{B_{mn} mn e_{22k} u_k^{(m)}\} \tag{3.32}$$
$$+ \left(\frac{n\pi}{2b}\right)^2 \epsilon_{22} \phi^{(n)} = 0$$

for $n = 1, 2, \ldots, N$

The forcing functions that are supposed to be given with a problem appear on the r.h.s.'s of (3.31) and (3.32). The conditions ensuring the uniqueness of the solution of the system of eqns (3.29), (3.30)–(3.9) and (3.11) are established in [8] through the development of a uniqueness theorem.

Formulation for crystal class C_{6v}

Here we focus attention on the special case of a hexagonal crystal in class C_{6v} with the poling direction along the x_2-axis. The 2-D displacement-electric potential field equations for this case are derived by substituting the arrays given in (2.18) into (3.31) and (3.32). The resulting differential system is as follows:

$$(1 + \delta_{no}) c_{11} u_{1,11}^{(n)} + \frac{1}{2}(1 + \delta_{no})(c_{11} - c_{13}) u_{1,33}^{(n)} + \frac{1}{2}(1 + \delta_{no})(c_{11} + c_{13})\, u_{3,13}^{(n)} + \frac{\pi}{2b} \sum_{m=0}^{N} \{D_{mn} u_{2,1}^{(m)}\}$$
$$- \frac{n\pi}{2b}(e_{21} + e_{16}) \phi_{,1}^{(n)} + \left(\frac{n\pi}{2b}\right)^2 c_{44} u_1^{(n)} - (1 + \delta_{no}) \rho \ddot{u}_1^{(n)} = \frac{-1}{b} F_1^{(n)}$$

for $n = 0, 1, \ldots, N$.

$$(1 + \delta_{no}) c_{44} (u_{2,11}^{(n)} + u_{2,33}^{(n)}) + \sum_{m=1}^{N} \{B_{mn} e_{16} (\phi_{,11}^{(m)} + \phi_{,33}^{(m)})\}$$
$$- \frac{\pi}{2b} \sum_{m=0}^{N} \{D_{nm} (u_{1,1}^{(m)} + u_{3,3}^{(m)})\} - \left(\frac{n\pi}{2b}\right)^2 c_{22} u_2^{(n)} + \left(\frac{\pi}{2b}\right)^2 \sum_{m=1}^{N} \{B_{nm} mn e_{22} \phi^{(m)}\}$$
$$- (1 + \delta_{no}) \rho \ddot{u}_2^{(n)} = \frac{-1}{b} F_2^{(n)} + \frac{n\pi}{2b} B_{no} \frac{1}{b} e_{22} B$$

for $n = 0, 1, \ldots, N$

$$\frac{1}{2}(1 + \delta_{no})(c_{11} + c_{13}) u_{1,13}^{(n)} + \frac{1}{2}(1 + \delta_{no})(c_{11} - c_{13}) u_{3,11}^{(n)}$$
$$+ (1 + \delta_{no}) c_{11} u_{3,33}^{(n)} + \frac{\pi}{2b} \sum_{m=0}^{N} \{D_{mn} u_{2,3}^{(m)}\} \tag{3.33}$$
$$- \frac{n\pi}{2b}(e_{21} + e_{16}) \phi_{,1}^{(n)} - \left(\frac{n\pi}{2b}\right)^2 c_{44} u_3^{(n)} - (1 + \delta_{no}) \rho \ddot{u}_3^{(n)} = \frac{-1}{b} F_3^{(n)}$$

for $n = 0, 1, \ldots, N$

$$\sum_{m=0}^{N} \{B_{nm} e_{16} (u_{2,11}^{(m)} + u_{2,33}^{(m)})\} - \epsilon_{11} (\phi_{,11}^{(n)} + \phi_{,33}^{(n)})$$
$$+ \frac{n\pi}{2b} (e_{16} + e_{21})(u_{1,1}^{(n)} + u_{3,3}^{(n)})$$
$$+ \left(\frac{\pi}{2b}\right)^2 \sum_{m=0}^{N} \{B_{mn} mn e_{22} u_2^{(m)}\} + \left(\frac{n\pi}{2b}\right)^2 \epsilon_{22} \phi^{(n)} = 0$$

for $n = 1, 2, \ldots, N$,

in which we have defined

$$D_{mn} = mB_{mn}c_{12} - nB_{nm}c_{44}.$$

Due to the fact that the trigonometric functions in the expansions (3.6) and (3.7) are either odd or even, the question of the uncoupling of extensional and flexural modes is somewhat simplified in the 2-D case. In fact, a little reflection will show that those components $u_j^{(n)}$ for which $n+j$ is odd represent extensional modes of motion and those for which $n+j$ is even represent flexural modes. Keeping this rule and the definition of B_{mn} in mind, we notice that the first, third, and fourth of (3.33), for even n and the second of (3.33) for odd n involve only extensional components whereas the opposite situation involves only flexural components. (In this context, we consider $\phi^{(n)}$ a flexural component when n is odd and an extensional component when n is even. This is in line with the results from the 3-D theory.) Thus, we see that the separation of extensional and flexural modes of motion that transpires in the 3-D theory is also reflected in this particular approximate 2-D formulation.

4. DISPERSION RELATIONS

In this section we investigate the dispersion relations for harmonic waves propagating in an infinite piezoelectric plate. The material of the plate, its poling direction and the coordinate system to which it is referred are as in Section 2. In addition, we specify for the present problem the boundary conditions of vanishing tractions and vanishing electric potential at the two faces:

$$\left.\begin{array}{l} T_{2j} = 0 \\ \phi = 0 \end{array} \text{at } x_2 = \pm b. \right\} \tag{4.1}$$

As was noted earlier, a solution for this problem deduced from the 3-D linear theory of piezoelectricity already exists[1]. We want to compare the dispersion curves calculated from this solution to those calculated from the approximate theories of various orders described in the preceding chapters. We seek to establish by this means the regions in the frequency-wave number plane within which a particular approximate theory can be considered useful.

We consider here as in Ref. [1] straight-crested waves with propagation vectors lying in the plane of the plate. We can then set up the coordinate system so that the propagation vector is along the x_1-axis and assume for the field variables harmonic dependence on x_1 and t.

Treating at first briefly the case of the shear horizontal (SH) waves, it is shown in [8] that the exact solution for the displacement component u_3, which is the only field variable involved in the SH motion, is given by

$$u_3 = A_3\, e^{i(\xi x - \omega t)} \cos\left[\frac{n\pi}{2}\left(1 - \frac{x_2}{b}\right)\right] \quad \text{for } n = 0, 1, 2, .. \tag{4.2}$$

We observe from (4.2) that each element of the series expansion for u_3 in (3.1) is a SH solution of the problem. Also, we find that both the 3-D field equation governing u_3 and the 2-D equation governing $u_3^{(n)}$ (third of (3.33) with $F_3^{(n)}$ and the dependence on x_3 eliminated) give the same dispersion relation for the nth branch of the SH dispersion curves:

$$\Omega = [\bar{c}_{55}z^2 + n^2] \tag{4.3}$$

where Ω, $\bar{c}_{55}$ and z stand for the dimensionless frequency, elastic constant and wave number, respectively. They are defined by:

$$\Omega = \sqrt{\left(\frac{\rho}{c_{44}}\right)}\frac{\omega}{(\pi/2b)},\ \bar{c}_{IJ} = \frac{c_{IJ}}{c_{44}},\quad z = \xi\Big/\left(\frac{\pi}{2b}\right). \tag{4.4}$$

Since the approximate and exact theories give the same result for the SH displacement component in the x_3-direction, it follows that we have to consider the displacement components in the x_1 and x_2 directions along with the electric potential, in order to determine the limits of the usefulness of the approximate theories.

The dispersion relations for the general Nth order theory are derived from the field equations (3.33). When the forcing terms $F_j^{(n)}$ and $B(t)$ are set to zero and the dependence on the x_3-coordinate of the variables is eliminated, the SH components $u_3^{(n)}$ uncouple as discussed before. The remaining set of 3 equations can be rewritten separately for extensional and flexural variables. It is convenient for this purpose to define two integers, R and S by:

$$R=\begin{cases} N/2 & \text{for even } N \\ (N-1)/2 & \text{for odd } N \end{cases}$$
$$S=\begin{cases} \dfrac{N}{2}-1 & \text{for even } N \\ (N-1)/2 & \text{for odd } N. \end{cases} \tag{4.5}$$

Excluding the 2-D SH displacement components in the x_3-direction, the extensional variables in the Nth order theory are:

$$u_1^{(2k)}, k=0,1,\ldots,R$$
$$\phi^{(2k)}, k=1,2,\ldots,R$$
$$u_2^{(2k+1)}, k=0,1,\ldots,S.$$

These $(2R+S+2)$ variables satisfy the following $(2R+S+2)$ equations which are merely the first, second and fourth of (3.33) rewritten for extensional variables:

$$\begin{cases} (1+\delta_{ko})c_{11}u_{1,11}^{(2k)}+\sum_{r=0}^{S}\left\{\frac{\pi}{2b}D_{(2r+1)(2k)}u_{2,1}^{(2r+1)}\right\} \\ -k\frac{\pi}{b}(e_{21}+e_{16})\phi_{,1}^{(2k)}-\left(\frac{k\pi}{b}\right)^2 c_{44}u_1^{(2k)} \\ -(1+\delta_{ko})\rho\ddot{u}_1^{(2k)}=-\frac{1}{b}F_1^{(2k)} \text{ for } k=0,1,\ldots,R \\ c_{44}u_{2,11}^{(2k+1)}+\sum_{r=0}^{R}\{B_{(2r)(2k+1)}e_{16}\phi_{,11}^{(2r)} \\ +\frac{\pi}{2b}D_{(2r)(2k+1)}u_{1,1}^{(2r)}+\left(\frac{\pi}{2b}\right)^2 B_{(2k+1)(2r)}e_{22}(2r)(2k+1)\phi^{(2r)}\} \\ -\left[\frac{(2k+1)\pi}{2b}\right]^2 c_{22}u_2^{(2k+1)}-\rho\ddot{u}_2^{(2k+1)}=-\frac{1}{b}F_2^{(2k+1)}+\frac{(2k+1)\pi}{2b}B_{(2k+1)0}\left(\frac{1}{b}\right)e_{22}B(t) \\ \quad \text{for } k=0,1,2,\ldots,S \\ \sum_{r=0}^{S}\{B_{(2k)(2r+1)}e_{16}u_{2,11}^{(2r+1)}\}-\epsilon_{11}\phi_{,11}^{(2k)} \\ +\frac{k\pi}{b}(e_{16}+e_{21})u_{1,1}^{(2k)}+\left(\frac{k\pi}{b}\right)^2\epsilon_{22}\phi^{(2k)} \\ +\left(\frac{\pi}{2b}\right)^2 e_{22}(2k)\sum_{r=0}^{S}\{(2r+1)B_{(2r+1)(2k)}u_2^{(2r+1)}\}=0 \end{cases} \tag{4.6}$$

for $k=1,2,\ldots,R.$

To find the extensional dispersion relations for the Nth order theory, we let

$$F_1^{(2k)}=F_2^{(2k+1)}=0 \text{ for all } k;\ B(t)=0$$
$$u_1^{(2k)}=A_1^{(2k)}\,e^{i(\xi x-\omega t)} \text{ for } k=0,1,\ldots,R$$
$$u_2^{(2k+1)}=-iA_2^{(2k+1)}\,e^{i(\xi x-\omega t)} \text{ for } k=0,1,\ldots,S \tag{4.7}$$
$$\phi^{(2k)}=-iA_4^{(2k)}\,e^{i(\xi x-\omega t)} \text{ for } k=1,2,\ldots,R.$$

When these equations are substituted into (4.6), one obtains a set of $(2R+S+2)$ linear equations in the $(2R+S+2)$ complex constants $A_1^{(2k)}$, $A_2^{(2k+1)}$ and $A_4^{(2k)}$. To ensure the existence of a non-trivial solution, the determinant of the matrix of the coefficients of these complex constants in the system is set to zero. The resulting determinantal equation can be represented in the following form after non-dimensionalization:

$$|\underset{\sim}{G}=\begin{vmatrix} \underset{\sim}{G}^{(1)} & \underset{\sim}{G}^{(6)} & \underset{\sim}{G}^{(5)} \\ \underset{\sim}{G}^{(6)T} & \underset{\sim}{G}^{(2)} & \underset{\sim}{G}^{(4)} \\ \underset{\sim}{G}^{(5)T} & \underset{\sim}{G}^{(4)T} & \underset{\sim}{G}^{(3)} \end{vmatrix}=0. \tag{4.8}$$

Here the matrix $\underset{\sim}{G}$ of which the determinant is set to zero is represented in terms of sub-matrices $\underset{\sim}{G}^{(1)}$ to $\underset{\sim}{G}^{(6)}$ and their transposed matrices. These sub-matrices are defined as follows:

(i) $\underset{\sim}{G}^{(1)}$ is an $(R+1)$ by $(R+1)$ diagonal matrix. The diagonal elements are given by:

$$G^{(1)}_{pp}=-(1+\delta_{p1})\bar{c}_{11}z^2-(2p-2)^2+(1+\delta_{p1})\Omega^2$$
$$p=1,2,\ldots,R+1, \text{ no summation on } p. \tag{4.9}$$

(ii) $G^{(2)}$ is an $(S+1)$ by $(S+1)$ diagonal matrix. Its diagonal elements are given by:

$$G^{(2)}_{pp}=-z^2-(2p-1)^2\bar{c}_{22}+\Omega^2 \tag{4.10}$$
$$p=1,2,\ldots,S+1, \text{ no summation on } p.$$

(iii) $\underset{\sim}{G}^{(3)}$ is an $R\times R$ diagonal matrix with diagonal elements given by:

$$G^{(3)}_{pp}=\bar{\epsilon}_{11}z^2+(2p)^2 \qquad p=1,2,\ldots,R, \text{ no summation on } p. \tag{4.11}$$

(iv) $\underset{\sim}{G}^{(4)}$ is an $(S+1)\times R$ matrix. Its elements are given by:

$$G^{(4)}_{pq}=-B_{(2q)(2p-1)}\bar{e}_{16}z^2+\bar{e}_{22}(2q)(2p-1)B_{(2p-1)(2q)}$$
$$p=1,2,\ldots,S+1; \quad q=1,2,\ldots,R \tag{4.12}$$

(v) $\underset{\sim}{G}^{(5)}$ is an $(R+1)\times R$ matrix. Its elements are defined by:

$$G^{(5)}_{(p+1)(p)}=-2p(\bar{e}_{21}+\bar{e}_{16})z \tag{4.13}$$

for $p=1,2,\ldots,R$; others zero.

(vi) $\underset{\sim}{G}^{(6)}$ is an $(R+1)\times(S+1)$ matrix. Its elements are given by:

$$G^{(6)}_{pq}=\bar{D}_{(2q-1)(2p-2)}z \tag{4.14}$$
$$p=1,2,\ldots,R+1; \; q=1,2,\ldots,S+1$$

where

$$\bar{c}_{ij}=\frac{c_{ij}}{c_{44}},\; \bar{D}_{pq}=\frac{D_{pq}}{c_{44}},\quad \bar{e}_{ij}=\frac{e_{ij}}{\sqrt{(\epsilon_{22}c_{44})}},\quad \bar{\epsilon}_{ij}=\frac{\epsilon_{ij}}{\epsilon_{22}}.$$

It may be noted that the matrix $\underset{\sim}{G}$ defined above through sub-matrices $\underset{\sim}{G}^{(1)}$ to $\underset{\sim}{G}^{(6)}$ is symmetric. Equation (4.8) represents the dispersion relation for extensional waves in the infinite piezoelectric plate as derived from the Nth order approximate theory.

The dispersion relation corresponding to the flexural components of the wave motion is found in the same way. Excluding again the SH displacements components in the x_3-direction, we list here the flexural variables for the Nth order theory:

$$u_1^{(2k+1)},\quad k=0,1,\ldots,S$$
$$u_2^{(2k)},\quad k=0,1,\ldots,R$$
$$\phi^{(2k+1)},\quad k=0,1,\ldots,S.$$

We rewrite the first, second and fourth of (3.33) in terms of the $(2s+R+3)$ flexural variables in this list to obtain the following set of $(2S+R+3)$ equations:

$$\left.\begin{aligned}
&c_{11}u^{(2k+1)}_{1,11}+\sum_{r=0}^{R}\left\{\frac{\pi}{2b}D_{(2r)(2k+1)}u^{(2r)}_{2,1}\right\}\\
&-\frac{(2k+1)\pi}{2b}(e_{21}+e_{16})\phi^{(2k+1)}_{,1}-\left[\frac{(2k+1)\pi}{2b}\right]^2c_{44}u_1^{(2k+1)}\\
&-\rho\ddot{u}_1^{(2k+1)}=-\frac{1}{b}F_1^{(2k+1)},\qquad k=0,1,\ldots,S\\
&(1+\delta_{ko})c_{44}u^{(2k)}_{2,11}+\sum_{r=0}^{S}\left\{B_{(2r+1)(2k)}e_{16}\phi^{(2r+1)}_{,11}\right.\\
&\left.-\frac{\pi}{2b}D_{(2k)(2r+1)}u^{(2r+1)}_{1,1}+\left(\frac{\pi}{2b}\right)^2e_{22}(2r+1)(2k)B_{(2k)(2r+1)}\phi^{(2r+1)}\right\}\\
&-\left(\frac{k\pi}{b}\right)^2c_{22}u_2^{(2k)}-(1+\delta_{ko})\rho\ddot{u}_2^{(2k)}=-\frac{1}{b}F_2^{(2k)},\qquad k=0,1,\ldots,R\\
&\sum_{r=0}^{R}\{B_{(2k+1)(2r)}e_{16}u^{(2r)}_{2,11}\}-\epsilon_{11}\phi^{(2k+1)}_{,11}+\frac{(2k+1)\pi}{2b}(e_{16}+e_{21})u^{(2k+1)}_{1,1}\\
&+\left(\frac{\pi}{2b}\right)^2e_{22}(2k+1)\sum_{r=1}^{R}\{(2r)B_{(2r)(2k+1)}u_2^{(2k)}\}\\
&+\left[\frac{(2k+1)\pi}{2b}\right]^2\epsilon_{22}\phi^{(2k+1)}=0,\qquad k=0,1,\ldots,S
\end{aligned}\right\}\tag{4.15}$$

The flexural dispersion relation is derived from these equations by first letting,

$$\begin{aligned}
&F_1^{(2k+1)}=F_2^{(2k)}=0\text{ for all }k,\\
&u_1^{(2k+1)}=A_1^{(2k+1)}\,e^{i(\xi x_1-\omega t)},\quad k=0,1,\ldots,S\\
&u_2^{(2k)}=-i\,A_2^{(2k)}\,e^{i(\xi x_1-\omega t)},\quad k=0,1,\ldots,R\\
&\phi^{(2k+1)}=-i\,A_4^{(2k+1)}\,e^{i(\xi x_1-\omega t)},\quad k=0,1,\ldots,S.
\end{aligned}\tag{4.16}$$

Equations (4.16) are now substituted into (4.15) and a set of $(2S+R+3)$ linear equations in $(2S+R+3)$ complex constants $A_1^{(2k+1)}$, $A_2^{(2k)}$ and $A_4^{(2k+1)}$ is obtained. As before, we set the determinant of the matrix of the coefficients to zero to obtain the result:

$$|\underset{\sim}{H}|=\begin{vmatrix}\underset{\sim}{H}^{(1)} & \underset{\sim}{H}^{(6)} & \underset{\sim}{H}^{(5)}\\ \underset{\sim}{H}^{(6)T} & \underset{\sim}{H}^{(2)} & \underset{\sim}{H}^{(4)}\\ \underset{\sim}{H}^{(5)T} & \underset{\sim}{H}^{(4)T} & \underset{\sim}{H}^{(3)}\end{vmatrix}=0.\tag{4.17}$$

This is the dispersion relation for the flexural wave motion as derived from the Nth order approximate theory. The matrix $\underset{\sim}{H}$ is defined through the submatrices $\underset{\sim}{H}^{(1)}$ to $\underset{\sim}{H}^{(6)}$ which are in turn defined as follows:

(i) $\underset{\sim}{H}^{(1)}$ is a $(S+1)\times(S+1)$ diagonal matrix with the diagonal elements:

$$H^{(1)}_{pp}=-\bar{c}_{11}z^2-(2p-1)^2+\Omega^2\tag{4.18}$$
$$p=1,2,\ldots,S+1,\text{ no summation on }p.$$

(ii) $\underset{\sim}{H}^{(2)}$ is a $(R+1)\times(R+1)$ diagonal matrix with the diagonal elements:

$$H^{(2)}_{pp} = -(1+\delta_{p1})z^2 - (2p-2)^2\bar{c}_{22} + (1+\delta_{p1})\Omega^2 \qquad (4.19)$$
$$p = 1, 2, \ldots, R+1, \text{ no summation on } p.$$

(iii) $\underset{\sim}{H}^{(3)}$ is a $(S+1)\times(S+1)$ diagonal matrix with diagonal elements:

$$H^{(3)}_{pp} = \bar{\epsilon}_{11}z^2 + (2p-1)^2 \qquad (4.20)$$
$$p = 1, 2, \ldots, S, S+1, \text{ no summation on } p.$$

(iv) $\underset{\sim}{H}^{(4)}$ is a $(R+1)\times(S+1)$ matrix defined by:

$$H^{(4)}_{pq} = -B_{(2q-1)(2p-2)}\bar{e}_{16}z^2 + \bar{e}_{22}(2q-1)(2p-2)B_{(2p-2)(2q-1)} \qquad (4.21)$$
$$p = 1, 2, \ldots, R, R+1; \quad q = 1, 2, \ldots, S, S+1.$$

(v) $\underset{\sim}{H}^{(5)}$ is a $(S+1)\times(S+1)$ diagonal matrix with diagonal elements:

$$H^{(5)}_{pp} = -(2p-1)(\bar{e}_{21} + \bar{e}_{16})z \qquad (4.22)$$
$$p = 1, 2, \ldots, S, S+1, \text{ no summation on } p.$$

(vi) $\underset{\sim}{H}^{(6)}$ is a $(S+1)\times(R+1)$ matrix defined by:

$$H^{(6)}_{pq} = \bar{D}_{(2q-2)(2p-1)}z \qquad (4.23)$$
$$p = 1, 2, \ldots, S, S+1; \; q = 1, 2, \ldots, R, R+1.$$

The matrix $\underset{\sim}{H}$ thus defined through sub-matrices $\underset{\sim}{H}^{(1)}$ to $\underset{\sim}{H}^{(6)}$, is observed to be symmetric like the corresponding matrix $\underset{\sim}{G}$ of the extensional dispersion relation.

For the flexural dispersion relation corresponding to first order approximate theory, we obtain from eqn (4.17) with $N = 1$:

$$|H| = \begin{vmatrix} \Omega^2 - \bar{c}_{11}z^2 - 1 & -\alpha A_{10}z & -(\bar{e}_{21} + \bar{e}_{16})z \\ -\alpha A_{10}z & 2(\Omega^2 - z^2) & -\alpha A_{10}\bar{e}_{16}z^2 \\ -(\bar{e}_{21} + \bar{e}_{16})z & -\alpha A_{10}\bar{e}_{16}z^2 & \bar{\epsilon}_{11}z^2 + 1 \end{vmatrix} = 0. \qquad (4.24)$$

We now determine the value of the correction factor α so that the lowest branch of the flexural dispersion curve does not intersect the real part of the z-axis. Thus we avoid the physically unfeasible result of negative frequency for long wavelengths. Setting $\Omega = 0$ in (4.24) and expanding, we obtain:

$$\begin{aligned} &z^6(2\bar{c}_{11}\bar{\epsilon}_{11} + \alpha^2A_{10}^2\bar{e}_{16}^2\bar{c}_{11}) + z^4[2\bar{c}_{11} + 2\bar{\epsilon}_{11} \\ &\quad + \alpha^2A_{10}^2\bar{e}_{16}^2 - \alpha^2A_{10}^2\bar{\epsilon}_{11} - 2\alpha^2A_{10}^2\bar{e}_{16}(\bar{e}_{16} + \bar{e}_{21}) \\ &\quad + 2(\bar{e}_{21} + \bar{e}_{16})^2] + z^2(2 - \alpha^2A_{10}^2) = 0. \end{aligned} \qquad (4.25)$$

Setting the coefficient of z^2 in (4.25) to zero gives:

$$\alpha = \frac{\pi}{2\sqrt{2}}. \qquad (4.26)$$

Then all the non-trivial roots of (4.25) are found to be purely imaginary

$$z_1 = 0; \; z_{2,3} = \pm i\left[\frac{\bar{c}_{11} + \bar{e}_{21}^2}{\bar{c}_{11}(\bar{\epsilon}_{11} + \bar{e}_{16}^2)}\right]^{1/2}.$$

From this point on, the value for α indicated in (4.26) will have been substituted in all results.

For the extensional dispersion relation corresponding to the first order approximate theory ($N = 1$), eqn (4.8) gives:

$$|\underset{\sim}{G}| = \begin{vmatrix} -2\bar{c}_{11}z^2 + 2\Omega^2 & \sqrt{(2)}\bar{c}_{12}z \\ \sqrt{(2)}\bar{c}_{12}z & \Omega^2 - z^2 - \bar{c}_{22} \end{vmatrix} = 0. \tag{4.27}$$

Now setting $N = 2$ in (4.17), we obtain the flexural dispersion relation derived from the second order approximate theory:

$$|\underset{\sim}{H}| = \begin{vmatrix} \Omega^2 - 1 - \bar{c}_{11}z^2 & -\sqrt{(2)}z & (2A_{21}\bar{c}_{12} - A_{12}z) & -(\bar{e}_{21} + \bar{e}_{16})z \\ -\sqrt{(2)}z & 2\Omega^2 - 2z^2 & 0 & -\sqrt{(2)}\bar{e}_{16}z^2 \\ (2A_{21}\bar{c}_{12} - A_{12})z & 0 & \Omega^2 - 4\bar{c}_{22} - z^2 & -A_{12}\bar{e}_{16}z^2 + 2\bar{e}_{22}A_{21} \\ -(\bar{e}_{21} + \bar{e}_{16})z & -\sqrt{(2)}\bar{e}_{16}z^2 & -A_{12}\bar{e}_{16}z^2 + 2\bar{e}_{22}A_{21} & \bar{\epsilon}_{11}z^2 + 1 \end{vmatrix} = 0 \tag{4.28}$$

Finally setting $N = 2$ in eqn (4.10) we obtain the extensional dispersion relation for the second order approximate theory:

$$|\underset{\sim}{G}| = \begin{vmatrix} 2(\Omega^2 - \bar{c}_{11}z^2) & 0 & \sqrt{(2)}\bar{c}_{12}z & 0 \\ 0 & \Omega^2 - 4 - \bar{c}_{11}z^2 & \frac{4}{3\pi}(4 - \bar{c}_{12})z & -2(\bar{e}_{21} + \bar{e}_{16})z \\ \sqrt{(2)}\bar{c}_{12}z & \frac{4}{3\pi}(4 - \bar{c}_{12}z) & \Omega^2 - \bar{c}_{22} - z^2 & -\frac{8}{3\pi}(\bar{e}_{16}z^2 + \bar{e}_{22}) \\ 0 & -2(\bar{e}_{21} + \bar{e}_{16})z & \frac{-8}{3\pi}(\bar{e}_{16}z^2 + \bar{e}_{22}) & \bar{\epsilon}_{11}z^2 + 4 \end{vmatrix} = 0. \tag{4.29}$$

The dispersion curves corresponding to the dispersion relations for the first and second order approximate theories are plotted together with the appropriate branches of the exact dispersion curves in Figs. 2–5 as follows:

Figure 2: Flexural exact and flexural zeroth and first order approximate curves.

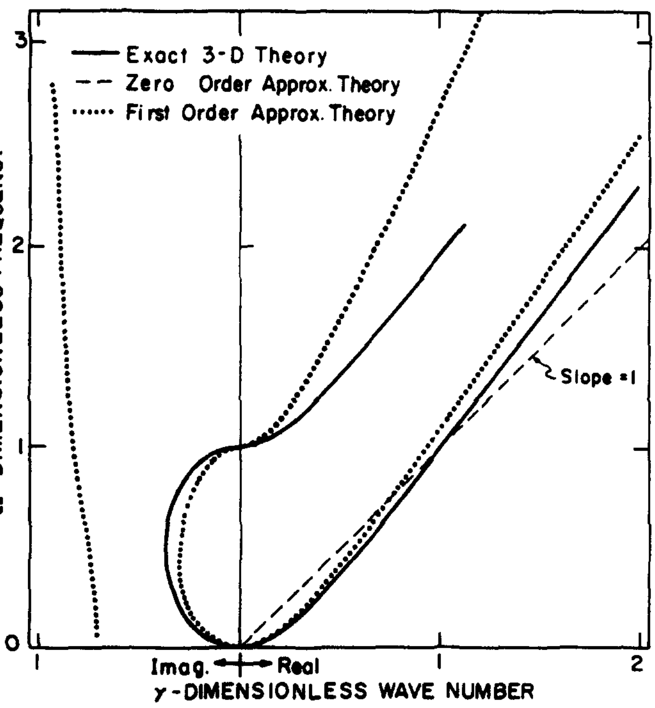

Fig. 2. Dispersion curves for flexural waves in PZT-5.

Figure 3: Flexural exact and flexural second order approximate curves.
Figure 4: Extensional exact and extensional first order approximate curves.
Figure 5: Extensional exact and extensional second order approximate curves.

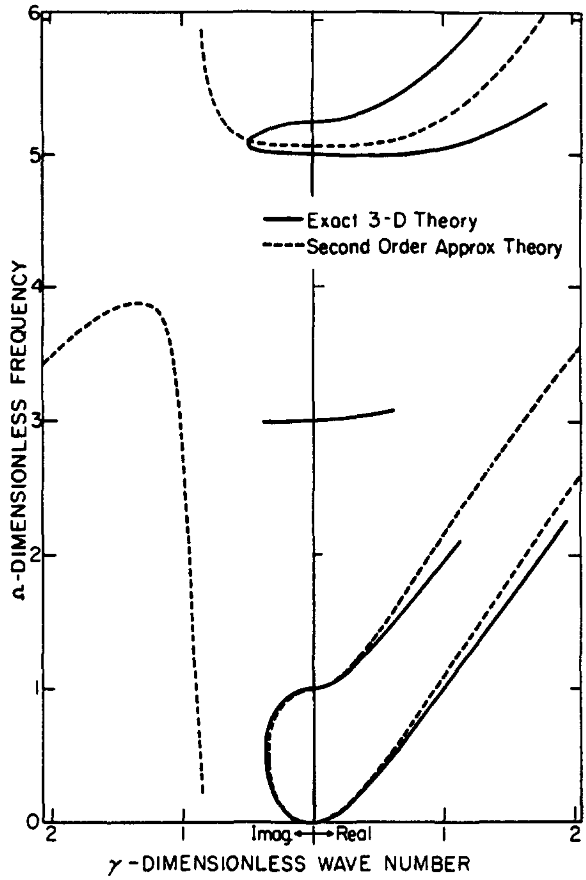

Fig. 3. Dispersion curves for flexural waves in PZT-5.

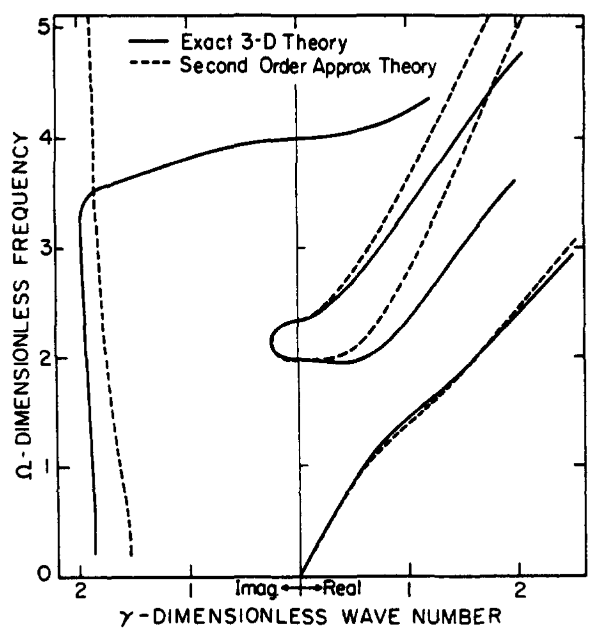

Fig. 4. Dispersion curves for extensional waves in PZT-5.

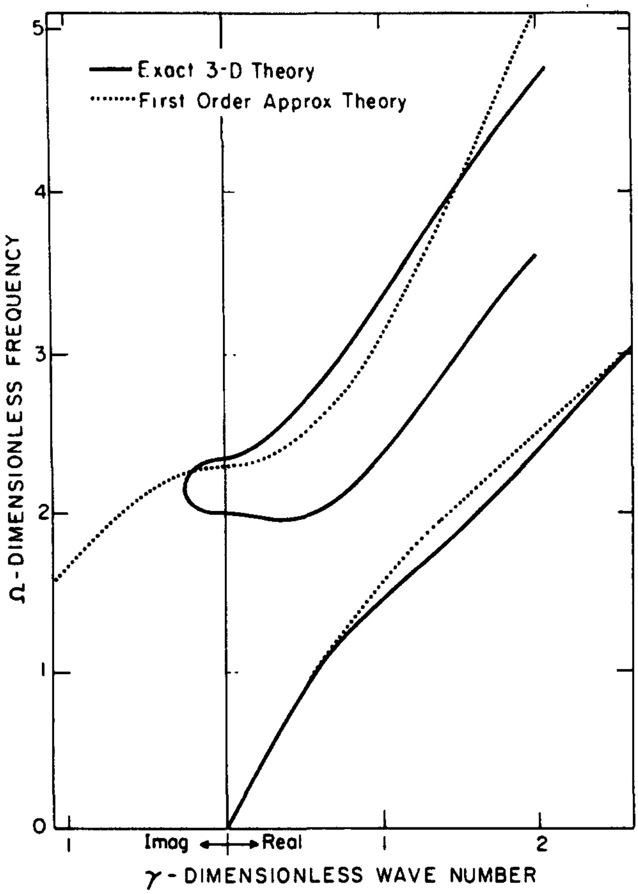

Fig. 5. Dispersion curves for extensional waves in PZT-5.

SUMMARY

The main result of this paper is a set of approximate 2-D theories useful for analyzing motion of piezoelectric layers when tractions and voltages at the faces of the layer are specified. Through such an approximate theory the dependence on the thickness coordinate is eliminated from the problem. These results permit the analysis of piezoelectric transducer response without making the usual 1-D assumptions, since the theories accommodate any functional dependence of the mechanical and electrical fields on the faces of the layer. Also the theory encompasses strong piezoelectric coupling of the type that occurs in usual transducer piezoelectrics such as PZT.

The theory was developed for arbitrary order N. The dispersion curves were calculated for orders up to $N = 2$ for the approximate theory and the corresponding branches were calculated from 3-D theory for PZT-5.

For applications involving motion at relatively high frequencies, the approximate theories presented here can be improved by introducing a second correction factor into the enthalpy density term (3.20). By means of this additional correction factor the slope of the dispersion curves for the approximate theories can be adjusted as the wave number gets larger so as to coincide with that for the surface wave velocity.

Acknowledgement—This work was sponsored by the National Science Foundation, Engineering Division, Solid Mechanics Program, Grant CME77-17150.

REFERENCES

1. H. F. Tiersten, Wave propagation in an infinite piezoelectric plate. *J. Acoust. Soc. Am.* 35(2), 234–239 (1963).
2. R. D. Mindlin, An introduction to the mathematical theory of the vibrations of elastic plates. U.S. Army Signal Corps Eng. Lab., Fort Monmouth, N.J. Signal Corps Contract DA-36-039 SC-56722 (1955).

3. P. C. Y. Lee and Z. Nikodem, An approximate theory for high-frequency vibrations of elastic plates. *Int. J. Solids Structures* **8**, 581–612 (1972).
4. Z. Nikodem and P. C. Y. Lee, Approximate theory of vibration of crystal plates at high frequencies. *Int. J. Solids Structures* **10**, 177–196 (1974).
5. S. Syngellakis and P. C. Y. Lee, An approximate theory for the high frequency vibrations on piezoelectric crystal plates. *Proc. 30th Ann. Symp. on Freq. Control*, pp. 184–190 (1976).
6. H. F. Tiersten, *Linear Piezoelectric Plate Vibrations*. Plenum Press, New York (1969).
7. D. A. Berlincourt, D. R. Curran and H. Taffe, Piezoelectric and piezomagnetic materials and their function in transducers. *Physical Acoustics*, Vol. I, Part A, *Methods and Devices* (Edited by W. P. Mason), pp. 169–270. Academic Press, New York (1964).
8. N. Bugdayci, A two-dimensional theory for piezoelectric layers used in electro-mechanical transducers. Ph.D. Thesis, University of California, Berkeley (1980).